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ON THE THEORY OF LIMIT LOADS*

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The theory of limit loads /1–7/ makes use of certain conditions imposed on the yield surface. It is usually assumed that the zero stresses lie strictly within it and(or) that the yield surface is bounded. These conditions, however, are not satisfied in some cases (e.g. for perfectly free-flowing media the yield surface has the form of a cone with the apex at zero /8/). Below it is shown that the basic assertions concerning the limit load remain valid also in many cases when the above conditions are not satisfied. However, the assertions will then not hold for all laods, but for a certain (wide) class described below. At the same time, the assertions may not be valid for other loads, e.g. the kinematic method yields only a trivial upper estimate for the load reserve coefficient. The corresponding example given below shows that the constraints usually imposed on the yield surface are essential from the point of satisfying the known assertions as applied to all loads.

1. First we shall consider, the simplicity, a discrete, rigorously perfect plastic system. Let $\mathbf{S} = \mathbf{R}^m$ be the space of internal forces, $\mathbf{F} = \mathbf{R}^m$ the space of the corresponding deformation rates and $\langle \boldsymbol{\sigma}, \mathbf{e} \rangle$ the intensity of the work done by internal forces $\boldsymbol{\sigma}$ on the deformation rates $\mathbf{e}$. The set C of plastically admissible forces (not emerging outside the yield surface) is defined by the relation $\Phi(\boldsymbol{\sigma}) \leqslant 0$ where Φ is a convex function. The force $\boldsymbol{\sigma}$ and the deformation rates $\mathbf{e}$ in the system are governed by the associated law or, which amounts to the same thing, by the principle of maximum plastic intensity $\langle \boldsymbol{\sigma}, \mathbf{e} \rangle \geqslant \langle \boldsymbol{\sigma}_*, \mathbf{e} \rangle$ for all $\boldsymbol{\sigma}_*$ satisfying the condition $\Phi(\boldsymbol{\sigma}_*) \leqslant 0$.

We assume that the forces $\boldsymbol{\sigma} = 0$ are plastically admissible ($\Phi(0) \leqslant 0$), but do not necessarily represent an internal point of the set C of admissible forces. The set is also not assumed to be bounded. Thus the conditions, sufficient /6/ for the static $\alpha(\mathbf{l})$ and kinematic $\beta(\mathbf{l})$ limiting load coefficients $\mathbf{l}$ to coincide, are not satisfied (we call the limiting coefficients the best estimates obtained with help of the static and kinematic coefficients /9/, respectively). Nor, generally speaking, is the condition (stating that $\boldsymbol{\sigma} = 0$ is an internal point of the set C) ensuring the non-deformability of the system under the load $\mu\mathbf{l}$ at $0 \leqslant \mu < \alpha(\mathbf{l})$ satisfied. Nevertheless, we shall describe a class of loads for which these properties are maintained (below we give an example of a load for which they are not maintained). The following assertion holds.

If a coefficient $\gamma > 0$ can be shown for a load $\mathbf{l}$ such, that the load $\gamma\mathbf{l}$ is balanced by the safe forces, then 1) the limiting static and dynamic coefficient of this load will be identical $\alpha(\mathbf{l}) = \beta(\mathbf{l})$, 2) the system will remain rigid under the load $\mu\mathbf{l}$ with coefficient $0 < \mu < \alpha(\mathbf{l})$.

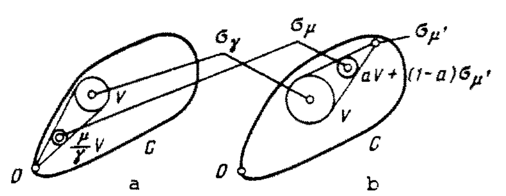

Fig.1

(The forces $\boldsymbol{\sigma}$ will be called safe if $\boldsymbol{\sigma}$ is an internal point of the set C of the physically admissible forces).

Let $\mathbf{s}_\gamma$ be safe forces balancing the load $\gamma\mathbf{l}$. We shall consider an auxiliary, rigorously perfect plastic system differing from the initial system only in the domain of plastically admissible forces which we shall define by the condition $\Phi^c(\boldsymbol{\sigma}) \equiv \Phi(\boldsymbol{\sigma} + \mathbf{s}_\gamma) \leqslant 0$.

We note that if the static coefficient of the load $\mathbf{l}$ $m_s \geqslant \gamma$ for the initial system, i.e. the load $m_s\mathbf{l}$ is balanced by some forces $\boldsymbol{\tau}$ and $\Phi(\boldsymbol{\tau}) \leqslant 0$, then the forces $\boldsymbol{\tau} - \mathbf{s}_\gamma$ balance the load $(m_s - \gamma)\mathbf{l}$ and are admissible for the auxiliary system; $\Phi^c(\boldsymbol{\tau} - \mathbf{s}_\gamma) = \Phi(\boldsymbol{\tau}) \leqslant 0$. Similarly, if m_s^c is the static coefficient of the load $\mathbf{l}$ for the auxiliary system then $m_s = m_s^c + \gamma$ will be its static coefficient for the initial system. Then the relation $\alpha = \alpha^c + \gamma$ will hold for the corresponding limiting static coefficients.

Let $\mathbf{e}$ denote the kinematically admissible deformation rates, in particular, and let the intensity of work done by the load $\mathbf{l} - \langle \gamma^{-1}\mathbf{s}_\gamma, \mathbf{e} \rangle \geqslant 0$ (the expression for the intensity is written remembering that the forces $\gamma^{-1}\mathbf{s}_\gamma$ balance the load $\mathbf{l}$). If $D(\mathbf{e}) = \sup\{\langle \boldsymbol{\sigma}, \mathbf{e} \rangle : \Phi(\boldsymbol{\sigma}) \leqslant 0\}$ is the

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dissipation /10/ for the initial system, then the dissipation for the auxiliary system will be

$$D^c(\mathbf{e}) = \sup\{\langle\sigma, \mathbf{e}\rangle : \Phi^c(\sigma) \leqslant 0\} = \sup\{\langle\tau - \mathbf{s}_\gamma, e\rangle : \Phi(\tau) \leqslant 0\} = D(\mathbf{e}) - \langle \mathbf{s}_\gamma, \mathbf{e}\rangle$$

Then the relation between the kinematic load coefficients for the initial (m_k) and auxiliary $(m_k{}^\circ)$ system becomes obvious

$$m_k = \frac{D(\mathbf{e})}{\langle\gamma^{-1}\mathbf{s}_\gamma, \mathbf{e}\rangle}, \quad m_k{}^\circ = \frac{D^c(\mathbf{e})}{\langle\gamma^{-1}\mathbf{s}_\gamma, \mathbf{e}\rangle} = m_k - \gamma$$

as well as the relation $\beta = \beta^c + \gamma$ connecting the corresponding limiting coefficients.

Since $\mathbf{s}_\gamma$ are the safe forces for the initial system it follows that $\sigma = 0$ is the internal point of the set of plastically admissible forces of the auxiliary system. This implies that conditions /6/ sufficient for the equality $\alpha^c = \beta^c$ to hold are satisfied for this system. By virtue of the relation established above, α and α°, β and β° are found at the same time as $\alpha = \beta$. This proves the first part of the assertion. Let $0 < \mu < \alpha(\mathbf{l})$. We shall show that the load $\mu\mathbf{l}$ can be balanced by certain safe forces σ_μ. When $0 < \mu \leqslant \gamma$, we take $\sigma_\mu = \mu\gamma^{-1}\sigma_\gamma$. Indeed, since σ_γ appears in the set C of permissible forces together with a certain neighbourhood V, σ_μ occurs in C together with the neighbourhood $\mu\gamma^{-1}V$ (see Fig.1a); the forces σ_μ are safe. When $\gamma < \mu < \alpha(l)$, we can find μ' $(\mu < \mu' < \alpha(\mathbf{l})$ such that the load $\mu' l$ will be balanced by certain admissible forces $\sigma_{\mu'}$. In this case we shall write μ in the form $\mu = a\gamma + (1-a)\mu'$ $(0 < a < 1)$ and take $\sigma_\mu = a\sigma_\gamma + (1-a)\sigma_{\mu'}$. It is clear that σ_μ are safe, since they appear in C together with the circle $aV + (1-a)\sigma_{\mu'}$ (Fig.1b). Thus both cases the load $\mu\mathbf{l}$ is balanced by the safe forces and the system therefore remains rigid /4/. This completes the proof.

Thus the basic assertion of the theory of limit laods remains valid for the class of loads shown. The reserve coefficient can be found by means of the kinematic or static methods, and the limiting estimates agree. For the loads belonging to the class in question the reserve coefficient $\alpha(\mathbf{l}) = \beta(\mathbf{l})$ retains its usual meaning. When $\mu > \alpha(\mathbf{l})$, the load $\mu\mathbf{l}$ cannot be

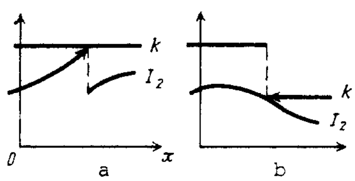

Fig.2

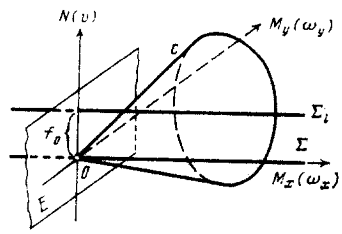

Fig.3

balanced by the plastically admissible forces, and when $0 < \mu < \alpha(\mathbf{l})$, it causes no deformation (failure). When $\mu = 0$, we have a unique situation, different from the usual one, namely the zero load is the limiting load. This will be the case for e.g. a perfectly granular medium. In this case the variation in the load applied to the system must be considered as starting not from the zero load, but from some load balanced by the safe stresses (e.g. for a perfectly granular medium this would be the hydrostatic pressure).

The proof of the assertion does not differ in any way, in the general case of a continuum, from the proof for the discrete case. We shall just explain certain concepts used. The finite dimensional space of internal forces is replaced in the case of a continuum by the functional space S of the stress fields (and similarly for the space of the rate of deformation $\mathbf{F}$). We assume that the conditions of Theorem 2 of /6/ hold for $S, \mathbf{F}$. The safe stress field is determined as above, i.e. it enters the set C of plastically admissible stress fields together with its neighbourhood. By the neighbourhood we mean a sphere of the stress space (if it is normal) or, generally, a neighbourhood in topology in which $\mathbf{F}$ is conjugate to S (for details see /6, 11/.

We note that the definition of safety of the stress field is somewhat more restricted than the usual one (in which the stress field σ is called safe if the inequality $\Phi_x(\sigma(x)) < 0$ where $\Phi_x(\sigma) = 0$ represents the yield condition corresponding to the point x. holds at every point of the body). Suppose, for example, in the case of the Mises plasticity condition $I_2(\sigma(x)) = k(x)$, the dependence of the second invariant of the stress deviator $I_2(\sigma)$ and yield point k on x, have the form shown in Fig.2a (the stress field has a discontinuity), or in Fig. 2b) the body is inhomogeneous and the yield point has a discontinuity). In both cases σ is safe in the usual sense, but not safe in the sense used here.

Note. The essential aspect of the proof of the second part of the assertion is not the fact that the stresses $\mathbf{s}_\gamma$ are safe, i.e. that they are in the set C of admissible stress fields together with their neighbourhoods, but the fact that they appear in C together with the set of the form $\mathbf{s}_\gamma + Q$ where Q is an absorbent set. (The set Q is called absorbent if for any

σ from S a number $\lambda > 0$ can be found such that $\lambda\sigma$ is in Q).

2. Let us indicate another type of condition ensuring, for any load, that its limiting static and kinematic coefficients are identical, namely the closure of the set $C + \Sigma$ where Σ is the collection of all selfequilibrating i.e. equilibrating the load $\mathbf{l} = 0$, stress fields belonging to S. Indeed, the closure of $C + \Sigma$ is equivalent to the closure of the set $C^{\wedge}$ introduced in /5/. In /6/ it was shown that the closure of $C^{\wedge}$ implies the equality $\alpha = \beta$ and the boundedness of C was used as its sufficient condition.

3. Using an example we shall now show what may happen if the restrictions imposed on the yield surface, usually adopted in the theory of limit laods, are not satisfied, and the applied load does not belong to the class shown in Sect.1. Namely, we shall describe a system and a load $\mathbf{l}$ applied to it. Although the load is a limit one, the kinematic method applied to it yields the value of the limiting kinematic coefficient equal to $\beta(\mathbf{l}) = +\infty$, or in other words, it does not provide the upper estimate for the limit load. In particular, in this case the limiting static and kinematic coefficients are no longer identical ($\alpha(\mathbf{l}) < +\infty$, $\beta(\mathbf{l}) = +\infty$). The lower estimate for the limit load derived with help of the statically admissible stresses also loses its meaning. In the case of the load $\mu(\mathbf{l})$ with the coefficient $0 < \mu < \alpha(\mathbf{l})$, we have the statically admissible stresses, nevertheless the load is a limit load and the body begins to flow (fail) under its influence.

Let us consider a beam rigidly clamped at one end and under tensile flexure caused by a longitudinal force and a moment, at the other end. Let x, y, z be the orthogonal coordinates with the z axis directed along the beam axis and the beam itself occupying the segment $0 \leqslant z \leqslant L$. Let us denote by $N(z), Q_x(z), Q_y(z), M_x(z), M_y(z)$ the longitudinal force, shear forces and bending moments respectively at the point z, and $v(z)$ and $\omega(z)$ the velocity at the point z and angular velocity of rotation of the corresponding transverse section.

The following conditions hold at the clamped beam end:

$$v(0) = 0, \quad \omega(0) = 0 \tag{3.1}$$

Let there be no transverse forces at the other beam end, and let the following kinematic condition hold:

$$Q_x(L) = 0, \quad Q_y(L) = 0; \quad \omega_x(L) = 0 \tag{3.2}$$

The beam is subjected to a longitudinal force f and bending moment about the axis $y - m_y$ both applied at the end $z = L$ of the beam. This means that at this end we have, in addition to (3.2), the conditions

$$M_y(L) = m_y, \quad N(L) = f \tag{3.3}$$

We note that since there are no external forces when $0 < z < L$ the equations of equilibrium of the beam and the first two conditions of (3.2) together yeild the equation $Q_x(z) = Q_y(z) = 0$. Then from the equations of equilibrium and conditions (3.3) we find, that $M_x = \text{const}$, $M_y = \text{const} = m_y$, $N = \text{const} = f$ (the system is statically indeterminate: the quantity M_x connected with the coupling reaction corresponding to the last condition of (3.2) is not determined). Thus, when the forces in the beam are in equilibrium with the load in question, they are characterized by three numbers M_x, M_y, N.

Example 1. Let us consider a discrete, strictly perfectly plastic system with internal forces $\sigma = (M_x, M_y, N)$ belonging to the space $\mathbf{S} = \mathbf{R}^3$, deformation rates $\mathbf{e} = (\omega_x, \omega_y, v)$ belonging to the space $\mathbf{F} = \mathbf{R}^3$, and the intensity of plastic work $\langle\sigma, \mathbf{e}\rangle = M_x\omega_x + M_y\omega_y + Nv$.

We write the kinematic conditions in the form

$$\omega_x = 0 \tag{3.4}$$

The conditions separate out the set $\mathbf{E}$ of the kinematically admissible deformation rates.

The conditions of equilibrium have the form $M_y = m_y$, $N = f$ (m_y, f is the given external load).

We consider, as the yield surface, the cone $M_xM_y = N^2$, $M_x \geqslant 0$ and the corresponding set of plastically admissible forces

$$C = \{(M_x, M_y, N) \in \mathbf{R}^3 : N^2 \leqslant M_xM_y, M_x \geqslant 0\}$$

Let a load $\mathbf{l}: f = f_0 > 0$, $m_y = 0$ act on the beam. Fig.3 shows the sets of plastically admissible forces C, kinematically admissible deformation rates $\mathbf{E}$, selfequilibrated forces Σ and the forces Σ_1 equilibrating the load $\mathbf{l}$.

Let us find the limiting kinematic load coefficient

$$\beta(\mathbf{l}) = \inf\{D(\mathbf{e})/(f_0v) : \mathbf{e} = (0, \omega_y, v), v > 0\} \tag{3.5}$$

Here the first addition indicates the general form of the deformation rate vector $\mathbf{e}$ kinematically admissible for the system in question, and the second condition is the condition of positive intensity of work done by the external forces $f_0v > 0$. The quantity $D(\mathbf{e})$ is given, as always, by the relation

$$D(\mathbf{e}) = \sup\{\langle\tau, \mathbf{e}\rangle : \tau \in C\} \tag{3.6}$$

We shall show that $\beta(\mathbf{l}) = -\infty$. To show this, it is sufficient to confirm that for the vector $\mathbf{e}$ shown in (3.5), the upper bound in (3.6) is equal to $+\infty$. Indeed, let us take, as the vector $\mathbf{e}$, the forces $\sigma_A = (2A(\omega_y/v)^2, A, A\sqrt{2}|\omega_y|/v)$ which is clearly plastically admissible.

Then $D(e) \geqslant \langle \sigma_A, e \rangle = A(\omega_y + \sqrt{2}|\omega_y|)$. When $\omega_y \neq 0$, the arbitrariness of $A > 0$ implies that $D(e) = +\infty$. If $\omega_y = 0$, an analogous conclusion is reached when $\sigma_A = (A, A, A)$.

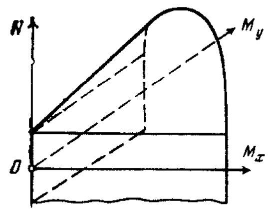

Fig.4

Thus we have $\beta(l) = +\infty$. In other words, if we attempt to use the kinematically admissible vector e (when the intensity of the work done by external forces is positive) to find the forces σ which would be connected with e by an associated law, we find that there are no such forces. This means that the kinematic method does not provide an estimate for the limit load.

Note. In the case of the load l in question, the load μl cannot be equilibrated by the plastically admissible forces no matter what the value of $\mu > 0$ is (Fig.3). However, this is not the reason for the impossibility of applying the kinematic method of assessment. The example given above can easily be altered so that the load would be balanced by the plastically admissible forces. To do this, it is sufficient to take the yield surface shown in Fig.4. Here we have, as before, $\beta(l) = +\infty$. We also note that in the case in question the set $C + \Sigma$ is closed.

Example 2. Let us consider the same beam with a given load as a continuous system. In this case the stresses and strain rates

$$\sigma = (M_x, M_y, N), \quad e = \left(\frac{d\omega_x}{\partial z}, \frac{d\omega_y}{dz}, \frac{dv}{dz}\right)$$

are functions in the segment $0 \leqslant z \leqslant L$; the intensity of the plastic work is

$$\langle \sigma, e \rangle = \int_0^L \left(M_x(z)\frac{d\omega_x}{dz} + M_y(z)\frac{d\omega_y}{dz} + N(z)\frac{dv}{dz}\right) dz$$

The dissipation $D(e) = \sup\langle \sigma, e \rangle$ where the upper limit is computed over all forces satisfying the condition $N^2(z) \leqslant M_x(z) M_y(z)$, $M_x(z) \geqslant 0$ with $0 \leqslant z \leqslant L$. In particular, we can take

$$\sigma = \sigma_A \equiv (M_{xA}, M_{yA}, N_A)$$

$$M_{xA}(z) = 2A\left(\frac{\omega_y(L)}{v(L)}\right)^2, \quad M_{yA} = A, \quad N(z) = A\sqrt{2}\frac{|\omega_y(L)|}{v(L)}$$

Then, since $\omega_x(L) = 0$, we have $D(e) \geqslant \langle \sigma_A, e \rangle = A(\omega_y(L) + \sqrt{2}|\omega_y(L)|)$ and, as in Example 1, $D(e) = +\infty$ and $\beta(l) = +\infty$, i.e. the kinematic method does not provide an estimate for the load reserve coefficient.

The examples given show that the assumptions adopted in /1–7/ or in Sect.1 and 2 above, are essential for the well-known assertions of the theory of limit loads to hold.

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